

Conformal Dynamics of Precursors to Fracture

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An exact integro-differential equation for the conformal map from the unit circle to the boundary of an evolving cavity in a stressed 2-dimensional solid is derived. This equation provides an accurate description of the dynamics of precursors to fracture when surface diffusion is important. The solution predicts the creation of sharp grooves that eventually lead to material failure via rapid fracture. Solutions of the new equation are demonstrated for the dynamics of an elliptical cavity and the stability of a circular cavity under biaxial stress, including the effects of surface stress.

PACS numbers: 46.50.+a, 62.20.Mk, 81.40.Np

The process of rapid fracture of solids which have failed to sustain stress is a poorly understood subject [1]; elasticity theory does not suffice to describe this process, since plastic deformations occur at the most interesting “process zone” where the actual fracture is taking place [2]. The displacement field is not the only relevant field, and even if the stress field is given everywhere, it is not known how the fracture propagates: there exist complex interactions with sound waves and maybe other fields [3, 4]. On the whole it is not obvious how to achieve a self consistent theory.

The situation is much clearer when one studies slow processes that may precede rapid fracture. In particular we will discuss in this Letter “precursors” to fracture. Namely, the dynamics of stress driven deformations of cavities (or free surfaces) in solids [5, 6, 7, 8, 9, 10, 11, 12, 13, 14]. Such deformations are expected to lead, eventually, to the creation of deep grooves which then suffer such large stresses that the solid fails via rapid fracture. The aim of this Letter is to derive and demonstrate a new equation of motion for the conformal map from the unit circle to the evolving stress-driven deformed cavity in 2-dimensional solids. This equation offers an accurate description of the slow evolution of the precursors to failure, until the moment that rapid fracture can be sustained. In contrast to all previous treatments we include both surface energy and surface stress and show that our equation is well posed.

To set up the problem imagine a 2-dimensional elastic medium with a hole of an arbitrary shape, whose boundary $z(s)$ is parametrized by the arc-length variable s . Boundary conditions at infinity load the medium, inducing a displacement field $u(\mathbf{r})$ with a strain tensor ϵ_{jk} , related to the stress by [15]

$$\sigma_{ij} \equiv \frac{E}{1+\nu} \left(\epsilon_{ij} + \frac{\nu}{1-2\nu} \delta_{ij} \sum_k \epsilon_{kk} \right). \quad (1)$$

Here ν is the Poisson ratio and E the Young modulus. Under stress there begins a process of surface diffusion which deforms the boundary, with dynamics determined

by the velocity v_n normal to the boundary,

$$v_n = -D \frac{\partial^2 \mu}{\partial s^2}, \quad (2)$$

where D is a diffusion coefficient that depends on the material and the temperature, and μ is the chemical potential at the boundary [16]. The chemical potential is obtained from the change in total energy associated with an infinitesimal variation of the interface [6, 17]:

$$\mu = \mu_0 + C[S - \gamma\kappa + \beta \left(\frac{\partial \epsilon_{tt}}{\partial n} - \kappa \epsilon_{tt} \right)], \quad (3)$$

where γ , κ and β are the surface energy, the mean curvature and surface stress respectively. S stands for the deformation energy $S = \sum_{i,j} \epsilon_{ij} \sigma_{ij} / 2$ [8]. The notation t and n stands for “tangent” and “normal” to the interface, defined precisely in Eqs.(6). μ_0 and C are the reference chemical potential and a material parameter that we can take as zero and unity respectively. Note that we differ from [7, 8, 10] in taking into account the surface stress. Ref. [18] incorporated all this physics but performed stability analysis of the flat interface only.

To evolve the cavity we need to compute then the chemical potential on its boundary, consistent with the evolving stress field in the medium. We will assert that on the slow time scale of surface diffusion the elastic medium is in equilibrium, i.e. $\sum_k \partial_k \sigma_{jk} = 0$ for all j . The general solution of these equations in 2-dimensional is given by [15]

$$\sigma_{xx} = \partial_y^2 \chi, \quad \sigma_{yy} = \partial_x^2 \chi, \quad \sigma_{xy} = -\partial_{xy} \chi, \quad (4)$$

where the so called Airy potential χ fulfills the biharmonic equation $\Delta^2 \chi = 0$. The general solution of this equation is written in complex notation, with $z = x + iy$, $\bar{z} = x - iy$, as

$$\chi(z, \bar{z}) = \Re \left[\bar{z} \phi(z) + \tilde{\psi}(z) \right], \quad (5)$$

where $\phi(z)$ and $\tilde{\psi}(z)$ are any pair of analytic functions, to be determined from the boundary conditions.

Consider the boundary of the cavity. With α the angle between the tangent and the x -axis at $z(s)$, define derivatives in the tangent and normal directions:

$$\begin{aligned}\partial_t &= \cos(\alpha) \partial_x + \sin(\alpha) \partial_y \\ \partial_n &= -\cos(\alpha) \partial_y + \sin(\alpha) \partial_x.\end{aligned}\quad (6)$$

The surface stress now must be balanced by the normal component of the stress [17, 18]:

$$\partial_{tt}\chi = \sigma_{nn} = \beta\kappa. \quad \text{on the crack.} \quad (7)$$

On the other hand the mixed derivatives vanish since there is no restoring force along the boundary,

$$-\partial_{tn}\chi = \sigma_{tn} = \sigma_{nt} = 0 \quad \text{on the crack.} \quad (8)$$

Using these boundary conditions we can evaluate the chemical potential μ on the boundary [In principle μ has terms that cannot be written in terms of $\text{Tr}\sigma$ alone, however these terms add to zero as a consequence of $\partial_n\sigma_{nn} = -\partial_t\sigma_{nt}$ and $d\sigma_{nt}/ds = 0$ on the boundary]

$$\mu = \left[\frac{1-\nu^2}{E} \left(\frac{(\text{Tr}\sigma)^2}{2} + \beta\partial_n\text{Tr}\sigma - \beta\kappa\text{Tr}\sigma \right) - \gamma\kappa \right] \quad (9)$$

Using the fact that $4\partial^2\chi/\partial z\partial\bar{z} = \sigma_{xx} + \sigma_{yy} = \sigma_{tt} + \sigma_{nn}$ we can immediately read from Eq. (5),

$$\text{Tr}\sigma = 4\Re[\phi'(z)]. \quad (10)$$

Thus to compute μ and its derivatives and advance the cavity we only need to determine the function $\phi(z)$. The boundary conditions (7) and (8) are expressed in terms of $\phi(z)$ and $\tilde{\psi}(z)$ by using Eqs. (6) -(8). We note that $(\cos\alpha + i\sin\alpha)(\partial_t - i\partial_n) = 2\partial_{\bar{z}}$ and on the boundary

$$\partial_t\partial_{\bar{z}}\chi = \frac{1}{2}(\cos\alpha + i\sin\alpha)\partial_{tt}\chi = \frac{\beta\kappa}{2}\partial_t z, \quad (11)$$

where we identify $\partial_t z(s)$ as the unit vector tangent to the boundary. Writing the mean curvature as $\kappa = \partial_t^2 z(s)/i\partial_t z(s)$, this condition reads $\partial_t\{\partial_{\bar{z}}\chi + i\beta\partial_t z(s)/2\} = 0$. Thus the boundary condition on the interface [19] is

$$\phi(z(s)) + z(s) \overline{\phi'(z(s)) + \tilde{\psi}(z(s))} = -i\beta\partial_t z(s) + K, \quad (12)$$

where $\tilde{\psi}(z) \equiv \tilde{\psi}(z)$ and K is a constant (that can be chosen zero with impunity).

For the boundary conditions at infinity we consider biaxial loading:

$$\sigma_{xx}(\infty) = \sigma_0, \quad \sigma_{yy}(\infty) = \sigma_0, \quad \sigma_{xy}(\infty) = 0. \quad (13)$$

It was shown in [19] that if the integral over the boundary of the RHS of Eq. (12) is zero, then the finiteness of the stresses at infinity implies that the analytic functions have a Laurent expansion of the form

$$\begin{aligned}\phi(z) &= \phi_1 z + \sum_{i=0} \phi_{-i} z^{-i} \\ \psi(z) &= \psi_1 z + \sum_{i=0} \psi_{-i} z^{-i}\end{aligned}\quad (14)$$

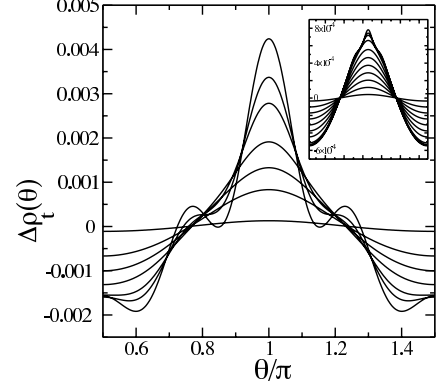


FIG. 1: Evolution of an ellipse under a biaxial load. Shown is the change in the radius $\Delta\rho_t(\theta) \equiv |\Phi(\epsilon, t)| - |\Phi(\epsilon, 0)|$ (in dimensionless units) relative to its initial value. The plot is given at dimensionless times $10^4 \times t = 4, 30, 40, 58.9, 71.8, 77.0$ and 82.3 . In the insert we show the analogous evolution for a cavity without surface stress.

The freedoms that we have in determining the Airy function (5) allow us to choose $\phi_0 = 0$ and ϕ_1 real. Then using the boundary conditions (13) at infinity, we find $\phi_1 = \sigma_0/2$, $\psi_1 = 0$. To proceed consider now a conformal map $\Phi(\omega, t)$ which maps the exterior of the unit circle $\epsilon = \exp(i\theta)$ to the exterior of boundary $z(s)$. Our central aim in this Letter is to derive an equation of motion of this map. The normal velocity v_n is related to the conformal map as follows: denote by $\hat{n} = n_x + in_y$ and $z(s) = x(s) + iy(s)$ the unit vector normal to the boundary, and the position of the boundary respectively. Then

$$v_n(s) = \dot{x}(s)n_x + \dot{y}(s)n_y = \Re \left[\frac{dz(s)}{dt} \bar{\hat{n}} \right] = \Re \left[\frac{d\Phi(\epsilon, t)}{dt} \bar{\hat{n}} \right]. \quad (15)$$

(Note that below we use ϵ and $\exp(i\theta)$ interchangeably). The tangent (\hat{t}) and normal unit vector are given in terms of the conformal map by

$$\begin{aligned}\hat{t} &= \frac{\partial\Phi(\epsilon)}{\partial s} = \frac{d\theta}{ds} \frac{\partial\Phi(\epsilon)}{\partial\theta} = \frac{1}{|\Phi'(\epsilon)|} \frac{\partial\Phi(\epsilon)}{\partial\theta} \\ \hat{n} &= -i\hat{t}.\end{aligned}\quad (16)$$

Using this in Eq. (15) and remembering that $d\Phi(e^{i\theta})/dt = \partial_t\Phi + \Phi'e^{i\theta}i\theta_t$ we derive:

$$v_n(s) = \Re \left(\partial_t\Phi(\epsilon) \epsilon \frac{\overline{\Phi'(\epsilon)}}{|\Phi'(\epsilon)|} \right). \quad (17)$$

We rewrite this equation,

$$\partial_t \Phi(\epsilon) \epsilon \frac{\overline{\Phi'(\epsilon)}}{|\Phi'(\epsilon)|} = v_n + iC, \quad (18)$$

with an unknown imaginary part C . Multiplying the last equation by $\epsilon \Phi' / |\Phi'|$ we get the equivalent equation

$$\partial_t \Phi = \epsilon \Phi'(\epsilon) \left(\frac{v_n}{|\Phi'|} + iC' \right) \quad (19)$$

and unknown C' . This equation, valid on the interface, can be analytically continued outside the unit circle. We need to choose C' such that the term in the parenthesis is an analytic function, removing the freedom in C' . This also fixes the parameterization which was so far arbitrary. All this is achieved with the Poisson integral formula,

$$\partial_t \Phi = \omega \Phi'(\omega) \int_0^{2\pi} \frac{d\theta}{2\pi} \frac{\omega + e^{i\theta}}{\omega - e^{i\theta}} \frac{v_n(e^{i\theta})}{|\Phi'(e^{i\theta})|} \quad (20)$$

The equation, being analytic, must have analytic solutions which provide the dynamics of the conformal map. In practice, the Poisson integral formula is best expressed in terms of the Fourier components of the function. Given a real function on the unit circle

$$g = \frac{v_n(e^{i\theta})}{|\Phi'(e^{i\theta})|} = a_0 + \sum_{n \geq 1} (a_n e^{in\theta} + \bar{a}_n e^{-in\theta}), \quad (21)$$

the analytic function outside the circle whose real part on the unit circle is g is

$$G = \int_0^{2\pi} \frac{d\theta}{2\pi} \frac{\omega + e^{i\theta}}{\omega - e^{i\theta}} g(e^{i\theta}) = a_0 + 2 \sum_{n \geq 1} \bar{a}_n \omega^{-n} \quad (22)$$

Thus finally the equation of the conformal map reads

$$\partial_t \Phi = \omega \Phi'(\omega) G(\omega). \quad (23)$$

The conformal map itself (which is univalent) has a Laurent expansion of the form

$$\Phi(\omega, t) = F_1(t) \omega + F_0(t) + \sum_{n=1}^{\infty} F_{-n}(t) \omega^{-n}. \quad (24)$$

By taking derivatives with respect to t and ω and substituting back in Eq.(23) we reduce the dynamics to an infinite set of ordinary differential equations for the Laurent coefficients $F_i(t)$. All that remains is to compute the function $g(\epsilon)$ that is given in terms of $v_n(\epsilon)$ and the conformal map. To compute the normal velocity $v_n(\epsilon)$ first compute $\text{Tr}\sigma$ on the boundary. Rewrite Eq.(12) in terms of the conformal map at time t :

$$\begin{aligned} \phi(\Phi(\epsilon, t)) + \frac{\Phi(\epsilon, t)}{\Phi'(\epsilon, t)} \overline{\phi'(\Phi(\epsilon, t))} + \overline{\psi(\Phi(\epsilon, t))} \\ = \beta \epsilon \frac{\Phi'(\epsilon, t)}{|\Phi'(\epsilon, t)|}. \end{aligned} \quad (25)$$

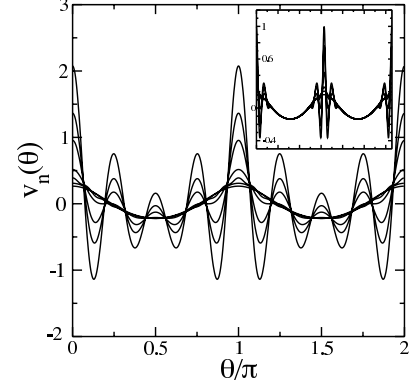


FIG. 2: The normal velocity as a function of the angle θ on the unit circle. The times are the same as in Fig. 1. The insert shows the analogous figure for a cavity without surface stress.

According to Eq.(24) the functions in Eq. (25) read

$$\begin{aligned} \phi(\Phi(\epsilon, t)) &= \phi_1 F_1(t) \epsilon + \phi_0(\epsilon) \\ \psi(\Phi(\epsilon, t)) &= \psi_1 F_1(t) \epsilon + \psi_0(\epsilon), \end{aligned} \quad (26)$$

where the only unknowns now are the expansion coefficients of the functions ϕ_0 and ψ_0 ,

$$\phi_0(\epsilon) = \sum_{n=0}^{\infty} u_{-n} \epsilon^{-n}; \quad \psi_0(\epsilon) = \sum_{n=0}^{\infty} v_{-n} \epsilon^{-n}. \quad (27)$$

At this point we need to compute the two Fourier transforms

$$\frac{\Phi(\epsilon, t)}{\Phi'(\epsilon, t)} = \sum_{n=-\infty}^{\infty} b_j \epsilon^j; \quad \frac{\Phi'(\epsilon, t)}{|\Phi'(\epsilon, t)|} = \sum_{n=-\infty}^{\infty} c_j \epsilon^j, \quad (28)$$

substitute them in to Eq. (25), and obtain a system of equations for the coefficients u_{-n} and v_{-n} . In fact upon the substitution one finds that the determination of the function $\phi(\omega)$ is independent of $\psi(\omega)$, requiring only negative powers of ϵ .

Having found the function $\phi(\omega)$ we employ Eq. (10) to compute the trace of the stress tensor anywhere on the boundary. The curvature κ is:

$$\kappa = \Re \left(\frac{1}{|\Phi'|} \left(1 + \frac{\Phi''}{\Phi'} e^{i\theta} \right) \right). \quad (29)$$

Taking the second derivative of the chemical potential we have $v_n(\epsilon)$. With this function at hand we can return to Eq. (21) and step the equations for the Laurent coefficients, finding the new conformal map, etc.

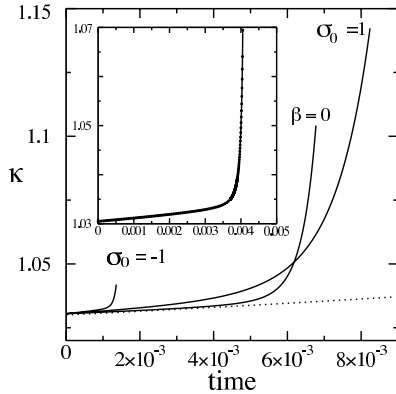


FIG. 3: The curvature at the tip as a function of time for an ellipse under compression and extension, and for zero surface stress. Dotted line: the curvature predicted by the linear stability analysis.

To demonstrate the efficiency of this procedure, and the interesting predictions it offers for the dynamics of cavities in stressed solids, we present (i) the evolution of

an initial elliptical hole and (ii) the stability analysis of a circular hole. For (i) our initial conformal map is

$$\Phi(\omega, t = 0) = F_1(0)\omega + \frac{F_{-1}(0)}{\omega}, \quad (30)$$

where we chose $F_1(0) = 1$, $F_{-1}(0) = 0.01$. For material parameters we took $(1 - \nu^2)/E = 1$, $\gamma = 0.2$, $D = 1$ and $\beta = 0.1$, all in dimensionless units. We study the evolution of the initial ellipse under biaxial load with $\sigma_0 = 1$. In Fig. 1 we show the time evolution of $\rho_t \equiv |\Phi(\epsilon, t)| - |\Phi(\epsilon, 0)|$. In the inset we show the analogous dynamics when $\beta = 0$, $\gamma = 0.1$. We see that the tip of the ellipse is advancing at the expense of two dips that develop on its side, but this effect is more dramatic for $\beta = 0$ as seen in the inset. In Fig. 2 we show the normal velocity, with the inset for the same parameters as in Fig. 1. Finally, in Fig. 3 we show the curvature at the tip. When the stress at the tip reached the Griffith criterion the material would yield by rapid fracture [20].

For (ii) we start with

$$\Phi(\omega, t) = R\omega + \sum_{n=1} f_n(t)\omega^{-n}; \quad |f_n| \ll 1 \quad \forall n, \quad (31)$$

which maps the unit circle ($\epsilon = e^{i\theta}$) to a wavy shaped circle of radius R . Using our equation (23) for the conformal map, and Eq. (25) which determines the stress, linearizing in small f_n , it is a straightforward calculation to obtain the stability eigenvalues in the form

$$\lambda_n = \frac{(n+1)^2}{R^2} \left\{ \frac{1-\nu^2}{E} \left[\left(\frac{8\sigma_0^2}{R} - \frac{12\beta\sigma_0}{R^2} \right) n - \left(\frac{4\beta^2}{R^3} + \frac{2\beta\sigma_0}{R^2} \right) n^2 - 2\frac{\beta^2}{R^3} n^3 \right] - \frac{2\gamma}{R^2} n - \frac{\gamma}{R^2} n^2 \right\}. \quad (32)$$

Contrary to the calculation in [18] our problem is well posed, the cubic term (n^3) is negative. The analytic result (32) is novel, showing precisely what load is needed to destabilize a circular cavity.

In summary, we stress that surface stress term in μ may matter: it introduces for sharp tips a competitive term proportional to $r^{-3/2}$ in contrast to the terms proportional to $1/r$ of the stress energy and the curvature. surface stress removes the degeneracy of compression and extension, as we see in Eq. (32). In the nonlinear regime

the grooves start to form faster for compression. These and related issues will be elaborated further in a future publication.

We thank Efim Brener for suggesting this problem to us, and for some useful e-discussions with him and with R. Spatschek. This work was supported in part by the Minerva Foundation, the European Commission under a TMR grant, and the Fundación Andes under project c-13760.

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